Gaussian effective potential for the standard model SU(2)×U(1) electroweak theory

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The Gaussian Effective Potential (GEP) is derived for the non-Abelian SU(2)×U(1) gauge theory of electroweak interactions. First the problem of gauge invariance is addressed in the Abelian U(1) theory, where an optimized GEP is shown to be gauge invariant. The method is then extended to the full non-Abelian gauge theory where, at variance with naive derivations, the GEP is proven to be a genuine variational tool in any gauge. The role of ghosts is discussed and the unitarity gauge is shown to be the only choice which allows calculability without insertion of further approximations. The GEP for the standard model is derived and its predictions are compared to the known phenomenology, thus showing that the GEP provides an alternative non-perturbative description of the known experimental data. By a consistent renormalization of masses the full non-Abelian calculation confirms the existence of a light Higgs boson in the non-perturbative strong coupling regime of the Higgs sector.

I. INTRODUCTION

It is now widely believed that the Higgs sector of electroweak interactions can be described by a scalar field with a self-interaction which could be large enough to raise some doubt on the validity of standard perturbative approaches. Thus, while perturbative results might be questioned, non-perturbative calculations would be required at least for comparison. Variational calculations are usually quite reliable for describing strong coupling phenomena, but their use in quantum field theory must face several difficult problems[1]. The problem of calculability can only be solved by use of a Gaussian wave functional, which has its merits as discussed in several papers on the Gaussian Effective Potential (GEP)[2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. An other important problem is the predominance of high momentum fluctuations in the vacuum espectation values. However the standard model of electroweak interactions is usually regarded as an effective model with a finite energy cut-off which regulates the theory. Thus the role of high momentum fluctuations is in part reduced, as the predictions of the GEP on effective models have been found to be reliable when compared to experimental results[12, 13, 14].

It has been pointed out that gauge invariance could be an other important challange for variational calculations, as there is no way to build a gauge invariant Gaussian functional in non-Abelian gauge theories[15]. It has been argued that in principle, if the states are not gauge invariant, they could be unphysical and span a larger Hilbert space where the unphysical energies could even be lower than the true physical vacuum[15]. However in this paper we show that a genuine variational GEP can be found for the non-Abelian SU(2)×U(1) standard model of electroweak interactions, and that for any chosen gauge the GEP can be proven to stay above the true effective potential. The genuine variational nature of the GEP makes the choice of gauge a question of taste and numerical convenience, and the physical unitarity gauge may be used without affecting the variational nature of the calculation.

Some further motivation for the work arises from a successful attempt to explain mass generation in the minimal left-right symmetric model of electroweak interactions[16, 17], where two scalar Higgs doublets and no bidoublet are present. At tree level that model predicts a vanishing expectation value for one of the scalar Higgs doublets, and that is a problem since all the fermionic masses turn out to be vanishing as well[18]. In that framework quantum fluctuations have been studied by the GEP and shown[16] to destabilize the symmetric vacuum towards a physical finite expectation value for both the Higgs doublets. While those findings are compatible with the phenomenology, their accuracy could be questioned for the neglection of all the weak couplings. Actually it was a simplified Abelian toy model, with only Higgs and fermionic fields. Thus an extension of the GEP method to the full non-Abelian SU(2)×U(1) gauge group would allow for quantitative predictions in the standard model and in its minimal left-right symmetric versions.

We must mention that this is not the first attempt to apply the GEP to the non-Abelian gauge theory, as previous naive calculations have been reported. It is very important to stress that the reliability of a variational calculation requires that no uncontrolled approximation should be added. The main result of this paper is the rigorous proof of the genuine variational nature of the GEP in unitarity gauge. In order to avoid problems regarding the gauge dependence

of the Hamiltonian, we derive the GEP in the Lagrangian formalism and start from a fully gauge invariant vacuum to vacuum transition amplitude. As in previous works on the U(1) theory[12, 13, 19], the GEP is derived by a systematic use of Jensen's inequality for expectation values of convex functions. As a consequence the GEP can never fall below the exact effective potential, and its minimum yields the best approximation to the vacuum energy density.

The derivation is useful for clarifying the role played by any gauge choice. In fact Jensen's inequality does not hold for Grassmann anticommuting fields and when ghost fields are present the naive use of the GEP turns out to be a tree level perturbative approximation. Thus the gauge must be properly chosen in order to avoid the presence of ghosts, and unitarity gauge turns out to be a good choice. Moreover we show that, even for the U(1) group, Jensen's inequality works at its best for some special gauge choice. In the non-Abelian theory the best gauge turns out to be the Unitarity one which also yields a very clear physical description.

In the standard model of weak interactions the method is shown to be a useful non-perturbative tool for the study of the Higgs sector in the strong coupling regime. The GEP provides a consistent renormalization of masses, and predicts the possible existence of a light Higgs boson even if the self-coupling were very large. In other words a light Higgs boson would not rule out a very large self-coupling which would question most of the perturbative calculations. The role of gauge interactions on the Higgs sector is also discussed and shown to be very small, as expected.

The paper is organized as follows: in section II the use of Jensen's inequality is discussed for the Abelian U(1) theory, and the resulting optimized GEP is shown to be gauge invariant; in section III the full non-Abelian SU(2)×U(1) gauge group is considered, and the main lines of the GEP derivation are outlined; in section IV the GEP is derived for the standard model of electroweak interactions, and in section V the gap equations are discussed in detail; section VI deals with a simple modified variational approach which allows a direct comparison with the phenomenology, while a consistent renormalization of masses is addressed in section VII where the phenomenological predictions are discussed for the strong-coupling regime of the Higgs sector.

II.   ABELIAN $U(1)$ THEORY

In the standard model of electroweak interactions the physical vacuum is believed to be at a broken symmetry minimum of the effective potential. Since the $SU(2) \times U(1)$ gauge symmetry is broken to the electromagnetic $U(1)$ group, the full gauge invariance of the GEP is not a real issue, provided that the method is shown to be a genuine variational calculation. However the electromagnetic $U(1)$ symmetry must remain unbroken in the physical vacuum and the method must give the same predictions for any choice of the unbroken $U(1)$ gauge. In this section we discuss the Abelian $U(1)$ theory and show how the GEP can be made invariant by a simple optimization of the variational approximation.

The GEP for the Abelian $U(1)$ theory (scalar electrodynamics) has been discussed by several authors[13, 19], and has been recently shown to provide a general interpolation of the experimental data for superconductors[12]. We briefly review the method in order to discuss its gauge invariance. In the Euclidean formalism the action reads

$$S = \int dx \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (D_\mu \phi)^* (D^\mu \phi) + \frac{1}{2} m^2 \phi^* \phi + \lambda (\phi^* \phi)^2 \right].$$ (1)

where ϕ is a complex (charged) scalar field, its covariant derivative is defined according to

$$D_\mu = \partial_\mu + ie A_\mu$$ (2)

and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The vacuum to vacuum transition amplitude is written as the functional integral

$$Z = \int D[\phi, \phi^*, A_\mu] e^{-S}.$$ (3)

In four space-time dimensions the action can be regarded as a toy model for the Higgs sector of standard electroweak interactions. In three dimensions the same action gives the static Ginzburg-Landau free-energy of a superconductor and Z plays the role of the partition function.

The action S in Eq.(1) has a local $U(1)$ symmetry as it is invariant for a local gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \chi(x)$$ (4)

$$\phi \to \phi \, e^{-ie\chi(x)}$$ (5)

where $\chi(x)$ is an arbitrary function. The integration over A_μ is then redundant in Eq.(3) and a gauge fixing term must be inserted according to the standard De Witt-Faddeev-Popov method[20]

$$Z = \int D[\phi, \phi^*, A_\mu] e^{-S} e^{-S_{fix}}. \tag{6}$$

where the gauge fixing action is

$$S_{fix} = \int dx \frac{1}{2\epsilon} f^2 \tag{7}$$

and $f(A, \phi) = 0$ is an arbitrary gauge constraint.

Z is invariant for any change of the parameter ϵ and of the constraint f. With some abuse of language, this invariance property is referred to as gauge invariance while it is a more general invariance since Z does not depend on the shape of the weight factor which has been added in Eq.(6). Only for $\epsilon \to 0$ the weight factor $\exp(-S_{fix})$ becomes a δ-function which enforces the constraint $f = 0$ on the fields. Thus standard gauge invariance denotes the invariance of the theory for any change of the constraint $f = 0$ in the limit $\epsilon \to 0$. This is a weaker condition, but unfortunately even this is not fulfilled by some approximations. In this paper we will make a distinction between the generalized gauge invariance and the standard gauge invariance, since it turns out that Jensen's inequality may break the first while leaving the second unbroken.

In fact let us take $f = (\partial_\mu A^\mu)$ and switch to polar coordinates $\phi \to \rho \exp(i\gamma)$ in the functional integral. The amplitude Z reads

$$Z = \int D[A_\mu, \rho^2] e^{-\int dx \mathcal{L}} \int D[\gamma] e^{-\int dx \mathcal{L}_\gamma} \tag{8}$$

where $\mathcal{L}$ is the phase independent Lagrangian

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \frac{1}{2} m^2 \rho^2 + \lambda \rho^4 + \frac{1}{2} e^2 \rho^2 A_\mu A^\mu + \frac{1}{2\epsilon} (\partial_\mu A^\mu)^2 \tag{9}$$

and $\mathcal{L}_\gamma$ is the sum of the Lagrangian terms which depend on the phase γ

$$\mathcal{L}_\gamma = \frac{1}{2} \rho^2 \partial_\mu \gamma \partial^\mu \gamma + e \rho^2 \partial_\mu \gamma A^\mu. \tag{10}$$

As a first step towards a gauge invariant GEP the phase γ is integrated by use of Jensen's inequality. While this integration has been sometimes regarded as exact[13, 21] it is not, but can be shown to be a genuine variational approximation[12]. In order to show that, let us denote by $\mathcal{L}_0$ the first term in the phase dependent Lagrangian Eq.(10)

$$\mathcal{L}_0 = \mathcal{L}_\gamma(e = 0) = \frac{1}{2} \rho^2 \partial_\mu \gamma \partial^\mu \gamma. \tag{11}$$

We observe that up to constant factors the exact integration over γ yields

$$\int D[\gamma] e^{-\int dx \mathcal{L}_0} \sim \prod_x \frac{1}{\rho}. \tag{12}$$

Thus we may write the $D[\rho^2]$ integral in Eq.(8) as

$$Z \sim \int D[A_\mu, \rho] e^{-\int dx \mathcal{L}} \left\{ \frac{\int D[\gamma] e^{-\int dx \mathcal{L}_\gamma}}{\int D[\gamma] e^{-\int dx \mathcal{L}_0}} \right\}. \tag{13}$$

An average over the phase can be defined as

$$\langle (...) \rangle_\gamma = \frac{\int D[\gamma] e^{-\int dx \mathcal{L}_0} (...)}{\int D[\gamma] e^{-\int dx \mathcal{L}_0}}. \tag{14}$$

and with this notation the exact Z amplitude Eq.(8) reads

$$Z = \int D[A_\mu, \rho] e^{-\int dx \mathcal{L}} \left\langle e^{-\int dx e \rho^2 \partial_\mu \gamma A^\mu} \right\rangle_\gamma.$$

(15)

Then the convexity of the exponential function (Jensen's inequality) ensures that

$$Z \geq \int D[A_\mu, \rho] \, e^{-\int dx \mathcal{L}} \, e^{-\int \left\langle e \rho^2 \partial_\mu \gamma A^\mu \right\rangle_\gamma dx}.$$

(16)

The average in the right hand side vanishes (it is linear in γ), and the approximate amplitude

$$Z_p = \int D[A_\mu, \rho] e^{-\int dx \mathcal{L}}$$

(17)

satisfies the variational constraint

$$Z \geq Z_p$$

(18)

so that the approximate potential $\mathcal{V}_p = -\ln Z_p$ is bounded by the exact one $\mathcal{V} = -\ln Z$

$$\mathcal{V}_p \geq \mathcal{V}.$$

(19)

Eventually the GEP may be evaluated by the same δ expansion method discussed in Ref.[19] and [22] and also reported by Ref.[13]. Inserting a source term for the real field ρ the generating functional $Z_p[J]$ reads

$$Z_p[J] = \int D[A_\mu, \rho] \, e^{-\int dx \mathcal{L}} \, e^{-\int dx J \rho}$$

(20)

and the effective potential is recovered by the Legendre transform

$$\mathcal{V}_p[\varphi] = -\ln Z_p + \int dx J \varphi$$

(21)

where φ is the average value of ρ. While the details of the derivation may be found in Ref.[12], we would like to discuss the gauge invariance of the result and the effects of any gauge change on the approximation.

Although Eq.(19) ensures that the approximate effective potential is bounded by the exact effective potential, a different choice of gauge could change the approximate result. Let us take in account the effects of a gauge change on the approximation in Eq.(16). A generic gauge change can be introduced by a shift of the constraint $f \to f' \neq \partial_\mu A^\mu$. After that we can always restore the constraint by a simple change of variables $A^\mu \to A^\mu + \partial_\mu \Lambda$ in the functional integration: the function Λ can be chosen in order to make $f' = \partial_\mu A^\mu$ again. However the change of variables adds some extra terms to the Lagrangian. Some of them only add constant or vanishing contributions (up to surface terms). Some others do not cancel but they only change the phase dependent Lagrangian $\mathcal{L}_\gamma$ which becomes

$$\mathcal{L}_\gamma = \frac{1}{2} \rho^2 \left(\partial_\mu \gamma + e \partial^\mu \Lambda \right)^2 + e \rho^2 \left(\partial_\mu \gamma + e \partial_\mu \Lambda \right) A^\mu.$$

(22)

It is obvious that we are allowed to restore the former approximate result by a simple shift of the phase $\gamma \to \gamma - e\Lambda$ before integrating. There is nothing wrong in that change as it is just an other change of variables in the functional integral. However we might want to keep the extra terms in $\mathcal{L}_\gamma$ and approximate the generating functional through Jensen's inequality according to Eq.(16). In that case we would get a different result as extra terms would remain in the Lagrangian. In other words the variational approximation seems to depend on our choice for the integration variable γ.

In fact the same ambiguity arises whenever we approximate a simple Gaussian integral by Jensen's inequality. For instance let us consider the exact result

$$I = \int_{-\infty}^{\infty} e^{-bx^2 + ax} dx = \sqrt{\frac{\pi}{b}} e^{\frac{a^2}{4b}}$$

(23)

where a and $b > 0$ are arbitrary real parameters. According to Eq.(14) we can define the average

$$\langle(...)\rangle_b = \frac{\int_{-\infty}^{\infty} e^{-bx^2}(...)}{\int_{-\infty}^{\infty} e^{-bx^2}}. \tag{24}$$

The integral can be approximated by use of Jensen's inequality

$$I = \int_{-\infty}^{\infty} e^{-bx^2} \langle e^{ax} \rangle_b dx \geq \sqrt{\frac{\pi}{b}} e^{\langle ax \rangle_b} = \sqrt{\frac{\pi}{b}}. \tag{25}$$

In fact this approximate result is smaller than the exact one, but it can be improved by the change of variable $x \to x + y$:

$$I = e^{-by^2+ay} \int_{-\infty}^{\infty} e^{-bx^2} \langle e^{(a-2by)x} \rangle_b dx \tag{26}$$

Of course the exact result does not change, but the approximate estimate does depend on y:

$$I \geq e^{-by^2+ay} \sqrt{\frac{\pi}{b}} e^{\langle (a-2by)x \rangle_b} = \sqrt{\frac{\pi}{b}} e^{-by^2+ay}. \tag{27}$$

The exponent $(-by^2+ay)$ has a maximum at the saddle point $y = a/(2b)$ where the approximate estimate reaches the exact result. It emerges that, in order to improve our variational method, the integration variable γ must be shifted before the use of Jensen's inequality. The best phase shift $\delta\gamma$ should be the saddle point of the modified $\mathcal{L}_\gamma$ Eq.(22), and should satisfy the linear equation

$$\partial_\mu(\delta\gamma + e\Lambda) = -eA_\mu. \tag{28}$$

Unfortunately in general there is no solution unless A^μ is a pure gauge. And this is the reason why phase integration does not give an exact result as it was claimed[13, 21]. The best we can do is to require that the phase change cancels the longitudinal component of the gauge field $e\partial_\mu\Lambda$ which has been inserted by the gauge change. That is equivalent to take $\delta\gamma = -e\Lambda$ which is exactly the same choice required in order to cancel all the effects of the gauge change in the Lagrangian $\mathcal{L}_\gamma$ Eq.(22).

Thus an optimized use of Jensen's inequality yields the very same approximate result for any choice of the gauge. In this sense we may state that the GEP obtained by this method is fully gauge invariant.

The whole discussion only makes sense in the limit $\epsilon \to 0$ when the gauge fixing term enforces the constraint $f = 0$ on the fields. For a general choice of ϵ there is no fixed gauge and the functional integration runs over the gauge group. This further averaging over the gauge group worsen the approximate result obtained by Jensen's inequality, and the result turns out to depend on ϵ. Of course the best choice is the limit $\epsilon \to 0$ since in that case the integration over the gauge group does not introduce any further approximation through the inequality which becomes exact.

III. NON ABELIAN $SU(2) \times U(1)$ THEORY

In this section we discuss the variational method for the full $SU(2) \times U(1)$ gauge group of electroweak interactions. The theory is described by the Euclidean Lagrangian

$$\mathcal{L} = \frac{1}{2}(D_\mu\Phi)^\dagger(D^\mu\Phi) + V(\Phi^\dagger\Phi) + \mathcal{L}_{YM} \tag{29}$$

where $\vec{A}_\mu$, B_μ are the gauge fields, $\mathcal{L}_{YM}$ is the Yang-Mill Lagrangian

$$\mathcal{L}_{YM} = \frac{1}{4}\vec{F}_{\mu\nu} \cdot \vec{F}^{\mu\nu} + \frac{1}{4}\left(\partial_\mu B_\nu - \partial_\nu B_\mu\right)^2 \tag{30}$$

in terms of the fields

$$\vec{F}_{\mu\nu} = \left(\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu\right) + g\vec{A}_\mu \times \vec{A}_\nu \tag{31}$$

and Φ is a Higgs doublet of complex fields ϕ_1, ϕ_2

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{32}$$

The covariant derivative reads

$$D_\mu = \left[\partial_\mu - ig\vec{A}_\mu \cdot \vec{T} + ig'B_\mu \frac{\mathbf{Y}}{2} \right] \tag{33}$$

where g, g' are the weak couplings and the generators are defined by the 2×2 matrices $\mathbf{Y} = 1$ and $\vec{T} = \frac{1}{2}\vec{\sigma}$. As usual the charge operator is $\mathbf{Q} = e(\mathbf{T}_3 + \mathbf{Y}/2)$.

In general the Higgs doublet Φ can be parametrized according to

$$\Phi = \rho e^{i\gamma} e^{i\sigma_3\phi} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{34}$$

where $\rho \geq 0$ is a real field, and the three phases γ, ϕ, θ may be taken in the ranges $0 \leq \gamma \leq 2\pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \theta \leq \pi/2$. Without fixing any special gauge we would like to discuss some general properties of the generating functional

$$Z[J] = \int D[\phi_1, \phi_2, \vec{A}, B] \, e^{-\int d^4x \, (\mathcal{L} - \rho J)}. \tag{35}$$

A change of integration variables yields

$$Z[J] = \int D[\rho^4, \sin^2\theta, \gamma, \phi, \vec{A}, B] \, e^{-\int d^4x \, (\mathcal{L} - \rho J)} \tag{36}$$

where $\mathcal{L}$ can be written as

$$\mathcal{L} = \mathcal{L}_\rho + \mathcal{L}_\mathcal{G} + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{YM} \tag{37}$$

according to the following definitions: $\mathcal{L}_\rho$ is the Lagrangian of the self-interacting scalar real field ρ

$$\mathcal{L}_\rho = \frac{1}{2}(\partial_\mu\rho)^2 + V(\rho^2); \tag{38}$$

$\mathcal{L}_\mathcal{G}$ contains the gauge phase quadratic terms

$$\mathcal{L}_\mathcal{G} = \frac{1}{2}\rho^2 \left[(\partial_\mu\gamma)^2 + (\partial_\mu\phi)^2 + (\partial_\mu\theta)^2 \right]; \tag{39}$$

$\mathcal{L}_2$ contains quadratic interaction terms for the gauge fields

$$\mathcal{L}_2 = \frac{1}{8}\rho^2 \left[g^2\vec{A}_\mu \cdot \vec{A}^\mu - 2gg'B^\mu\vec{A}_\mu \cdot \vec{R} + g'^2 B_\mu B^\mu \right]; \tag{40}$$

where $\vec{R}$ is the phase dependent vector

$$\vec{R} = \begin{pmatrix} \sin(2\theta)\cos(2\phi) \\ -\sin(2\theta)\sin(2\phi) \\ \cos(2\theta) \end{pmatrix}; \tag{41}$$

$\mathcal{L}_1$ contains linear interaction terms for the gauge fields

$$\mathcal{L}_1 = \frac{1}{2}\rho^2 g\vec{A}^\mu \cdot \vec{\Gamma}_\mu + \frac{1}{2}\rho^2 g'B^\mu\Theta_\mu \tag{42}$$

where $\vec{\Gamma}_\mu$ and Θ_μ depend on phases and are defined as follows

$$\vec{\Gamma}_\mu = \vec{R}\partial_\mu\gamma + \begin{pmatrix} \sin(2\phi)\partial_\mu\theta \\ \cos(2\phi)\partial_\mu\theta \\ -\partial_\mu\phi \end{pmatrix} \tag{43}$$

$$\Theta_\mu = \partial_\mu\gamma + \cos(2\theta)\partial_\mu\phi. \tag{44}$$

According to the standard De Witt-Faddeev-Popov method[20] the integration over the gauge group can be dealt with by insertion of a gauge fixing term

$$\mathcal{L}_{fix} = -\frac{1}{\epsilon}(f_\alpha)^2 \tag{45}$$

where the index α runs over the four gauge fields

$$f_\alpha = (\vec{f}, f_B). \tag{46}$$

The gauge invariance of the generating functional $Z[J]$ is preserved provided that a factor is also inserted in the integrand, equal to the determinant of the matrix

$$\mathcal{F}_{\alpha,\beta} = \left(\frac{\delta f_\alpha}{\delta\lambda_\beta}\right)_{\lambda_\beta=0} \tag{47}$$

where λ_β is the generic parameter of a gauge transformation[23]. The gauge invariant generating functional now reads

$$Z[J] = \int D[\rho^4, \sin^2\theta, \gamma, \phi, \vec{A}, B] \, \mathrm{Det}\mathcal{F} \, e^{-\int d^4x \, (\mathcal{L} + \mathcal{L}_{fix} - \rho J)}. \tag{48}$$

From a formal point of view the determinant can be seen as

$$\mathrm{Det}\mathcal{F} = e^{-\int d^4x \mathcal{L}_{gh}} \tag{49}$$

where the *ghost* Lagrangian $\mathcal{L}_{gh}$

$$\mathcal{L}_{gh} = -\mathrm{Tr}\log\mathcal{F} \tag{50}$$

can be written in terms of anticommuting Grassmann ghost fields. Thus the definition of the generating functional $Z[J]$ in Eq.(35) can be made gauge invariant by the replacement $\mathcal{L} \to \mathcal{L} + \mathcal{L}_{fix} + \mathcal{L}_{gh}$.

We would like to extend the variational method discussed in the previous section, and see how it works for the non-Abelian model. In analogy to Eq.(12) we can see that integration over phases yields

$$\int D[\sin^2\theta, \gamma, \phi] e^{-\int d^4x \mathcal{L}_\mathcal{G}} \sim \prod_x \frac{1}{\rho^3}. \tag{51}$$

Let us use the shorthand notation $D_\gamma = D[\sin^2\theta, \gamma, \phi]$ and $D_\rho = D[\rho, \vec{A}, B]$, and define the average over phases

$$\langle(...)\rangle_\gamma = \frac{\int D_\gamma e^{-\int d^4x\mathcal{L}_\mathcal{G}}(...)}{\int D_\gamma e^{-\int d^4x\mathcal{L}_\mathcal{G}}}. \tag{52}$$

The generating functional then reads

$$Z[J] = \int D_\rho \, e^{\int d^4x\rho J} \left\langle e^{-\int d^4x(\mathcal{L} + \mathcal{L}_{fix} + \mathcal{L}_{gh} - \mathcal{L}_\mathcal{G})} \right\rangle_\gamma. \tag{53}$$

Moreover for any trial Gaussian Lagrangian $\mathcal{L}_{GEP}(\rho, \vec{A}, B)$ which does not depend on the phases θ, ϕ, γ, a further average can be defined

$$\langle(...)\rangle_\rho = \frac{\int D_\rho e^{-\int d^4x \mathcal{L}_{GEP}}(...)}{\int D_\rho e^{-\int d^4x \mathcal{L}_{GEP}}}. \tag{54}$$

and the exact gauge invariant generating functional can be written as a double average

$$Z[J] = \left\langle e^{\int d^4x \rho J} \left\langle e^{-\int d^4x (\mathcal{L} + \mathcal{L}_{fix} + \mathcal{L}_{gh} - \mathcal{L}_{\mathcal{G}} - \mathcal{L}_{GEP})} \right\rangle_\gamma \right\rangle_\rho Z_0 \tag{55}$$

where

$$Z_0 = \int D_\rho e^{-\int d^4x \mathcal{L}_{GEP}}. \tag{56}$$

A variational approximation for the effective potential follows from the use of Jensen's inequality: the approximate generating functional $Z_{GEP}[J]$ is bound by the exact one as

$$Z[J] \geq Z_{GEP}[J] = Z_0\, e^{-\int d^4x \left\langle \langle \mathcal{L} + \mathcal{L}_{fix} + \mathcal{L}_{gh} - \mathcal{L}_{\mathcal{G}} - \mathcal{L}_{GEP} - J\rho \rangle_\gamma \right\rangle_\rho}. \tag{57}$$

Up to a total volume factor, the exact effective potential is defined as the Legendre transform

$$\mathcal{V}[\bar\rho] = -\log Z[J] + \int d^4x J\bar\rho \tag{58}$$

where $\bar\rho$ is the expectation value of the field ρ in the presence of the source J. We assume that $\langle\rho\rangle_\rho = \bar\rho$ where $\bar\rho$ is a parameter of the trial Lagrangian $\mathcal{L}_{GEP}$. In other words $\bar\rho$ is the central value of the quadratic Lagrangian $\mathcal{L}_{GEP}$. It follows that

$$\mathcal{V}[\bar\rho] \leq \mathcal{V}_{GEP}(\bar\rho) = -\log Z_{GEP}[J] + \int d^4x J\bar\rho \tag{59}$$

Thus the approximate Gaussian effective potential $\mathcal{V}_{GEP}$ is a genuine variational approximation of the exact effective potential, and can be evaluated by the double average

$$\mathcal{V}_{GEP}(\bar\rho) = -\log \int D_\rho e^{-\int d^4x \mathcal{L}_{GEP}} + \int d^4x \left\langle \langle \mathcal{L} + \mathcal{L}_{fix} + \mathcal{L}_{gh} - \mathcal{L}_{\mathcal{G}} - \mathcal{L}_{GEP} \rangle_\gamma \right\rangle_\rho. \tag{60}$$

The present derivation holds for any gauge choice, that means the method can be improved by a gauge change. In fact, as for the Abelian $U(1)$ theory, the limit $\epsilon \to 0$ should be imposed on $\mathcal{L}_{fix}$ in order to improve the reliability of Jensen's inequality in Eq.(57). Under that limit the integration over the gauge group does not introduce new approximations as the constraint in $\mathcal{L}_{fix}$ yields a δ function and the integration over the gauge group becomes exact (it is not affected by the inequality). On the other hand a gauge choice should not be a problem as the gauge symmetry is broken anyway in the physical vacuum.

The physics of the non-Abelian $SU(2) \times U(1)$ model is more evident in unitarity gauge which seems to be the natural choice for dicussing the symmetry breaking mechanism. However there is a more formal motivation for that choice which has to do with calculability. Provided that we take a simple quadratic shape for the trial Lagrangian $\mathcal{L}_{GEP}$, the Gaussian integral and the averages in Eq.(60) can be all easily evaluated with the important exception of the ghost term $\langle \mathcal{L}_{gh} \rangle$. The existence of this term makes the method useless since we do not know how to calculate its average. In a naive approach we could write $\mathcal{L}_{gh}$ in terms of anticommuting Grassmann ghost fields, but Jensen's inequality cannot be proven for Grassmann variables and the result would not be a genuine variational approximation. There would be no control on the approximation. An other naive approach would consist in the mere neglection of this term, and that can be shown to be the tree-level approximation of a perturbative expansion.

However in unitarity gauge the constraint functions f_α do not depend on the gauge fields: the mass of the ghost fields scales like $\epsilon^{-1/2}$ and becomes infinite in the $\epsilon \to 0$ limit, decoupling the ghosts from the physical fields. In other words the factor $\mathrm{Det}\mathcal{F}$ in Eq.(48) becomes a constant and can be carried out of the integral. Thus in unitarity gauge

the average of $\mathcal{L}_{gh}$ is a constant and can be neglected. We conclude that calculability makes the choice of unitarity gauge the only viable choice.

It is instructive to study the behaviour of $\mathcal{L}_{gh}$ in the renormalizable ξ-gauge of Fujikawa, Lee and Sanda[24] which is equivalent to the unitarity gauge in the $\epsilon = 1/\xi \to 0$ limit. The matrix $\mathcal{F}$ can be written as[23]

$$\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_{int} \tag{61}$$

where $\mathcal{F}_{int}$ contains a linear coupling with the gauge fields, $\mathcal{F}_0$ is the matrix

$$(\mathcal{F}_0)_{\alpha x, \beta y} = \left[\delta_{\alpha\beta} \partial_\mu \partial^\mu + \frac{1}{\epsilon} M_{\alpha\beta} \right] \delta^4(x - y) \tag{62}$$

and $M_{\alpha\beta}$ is a constant mass matrix. Inserting the definition Eq.(50) in Eq.(60), the double average of $\mathcal{L}_{gh}$ can be written as

$$\langle\langle \mathcal{L}_{gh} \rangle\rangle = -\langle\langle \mathrm{Tr} \log \mathcal{F}_0 \rangle\rangle - \langle\langle \mathrm{Tr} \log(1 + \mathcal{F}_0^{-1} \mathcal{F}_{int}) \rangle\rangle \tag{63}$$

The second term can be expanded yielding the perturbative series

$$\langle\langle \mathrm{Tr} \log(1 + \mathcal{F}_0^{-1} \mathcal{F}_{int}) \rangle\rangle \approx \mathrm{Tr} \langle\langle \mathcal{F}_0^{-1} \mathcal{F}_{int} \rangle\rangle - \frac{1}{2} \mathrm{Tr} \langle\langle \mathcal{F}_0^{-1} \mathcal{F}_{int} \mathcal{F}_0^{-1} \mathcal{F}_{int} \rangle\rangle + \dots \tag{64}$$

According to Eq.(62) the average $\langle\langle \mathcal{F}_0^{-1} \rangle\rangle$ can be regarded as the propagator for a massive particle (a ghost) whose mass scales like $1/\sqrt{\epsilon}$. The interaction vertex $\mathcal{F}_{int}$ is linear in the gauge fields, and the average of any pair $\langle\langle \mathcal{F}_{int} \mathcal{F}_{int} \rangle\rangle$ yields a gauge field propagator. Thus a diagrammatic expansion is recovered by Wick's theorem: Eq.(64) can be regarded as the sum of loop diagrams each consisting of a closed ghost ring crossed by any number of gauge lines. At tree-level, neglecting all the interaction lines, the double average of $\mathcal{L}_{gh}$ becomes a constant and can be neglected in the effective potential Eq.(60). Thus the naive neglection of $\mathcal{L}_{gh}$ is equivalent to the tree-level approximation of the perturbative expansion. However, in the $\epsilon \to 0$ limit, the ghost mass becomes infinite and all the terms in the expansion vanish. In the $\epsilon \to 0$ limit the renormalizable ξ-gauge becomes the unitarity gauge, and we recover the result that $\mathcal{L}_{gh}$ can only be neglected in the unitarity gauge.

With that gauge choice understood, the double average in Eq.(60) becomes trivial and the GEP can be easily evaluated provided that a simple quadratic shape is chosen for $\mathcal{L}_{GEP}$. Moreover if $\mathcal{L}_{GEP}$ is an even functional the double average of $\mathcal{L}_1$ also vanishes. However, we have seen in the previous section that, in order to get the best approximation from Jensen's inequality, the linear term must be shifted. In the Abelian $U(1)$ model the best choice was the transverse gauge fixed by the constraint $\partial_\mu A^\mu = 0$. In Unitarity gauge we still have a free overall electromagnetic $U(1)$ phase, and the best approximation arises from the transverse electromagnetic gauge. In order to show that, we must take a shift of the integration variables before taking the average. A linear change of variables is required first from the gauge fields $\vec{A}_\mu$, B_μ to the physical fields $W_\mu^\pm$, Z_μ, A_μ; then the best shift for the electromagnetic phase can be discussed, and eventually the double average will be taken.

IV. GEP FOR THE STANDARD MODEL

In the unitarity gauge ($\theta = \pi/2$) the physical massive gauge fields $W^\pm$, Z and the electromagnetic gauge field A are defined according to the linear transformation

$$A_\mu^1 = \frac{W_\mu^+ + W_\mu^-}{\sqrt{2}} \tag{65}$$

$$A_\mu^2 = \frac{W_\mu^+ - W_\mu^-}{i\sqrt{2}} \tag{66}$$

$$A_\mu^3 = \frac{e}{g} A_\mu - \frac{e}{g'} Z_\mu \tag{67}$$

$$B_\mu = -\frac{e}{g} Z_\mu - \frac{e}{g'} A_\mu \tag{68}$$

where the electromagnetic charge e follows from the constraint

$$\frac{e^2}{g^2} + \frac{e^2}{g'^2} = 1.$$

(69)

Insertion of these definitions in the quadratic Lagrangian term Eq.(40) yields

$$\mathcal{L}_2 = \frac{\rho^2}{v^2} M_W^2 W_\mu^+ W^{-\mu} + \frac{1}{2}\frac{\rho^2}{v^2} M_Z^2 Z_\mu Z^\mu$$

(70)

where the masses M_W and M_Z are given by the standard definitions

$$M_W = \frac{vg}{2}$$

(71)

$$M_Z = \frac{1}{2}v\sqrt{g^2 + g'^2}$$

(72)

in terms of the free parameter v. The gauge field A_μ remains massless, as it must be, since the electromagnetic $U(1)$ symmetry is unbroken. According to the discussion of Section II, for the $U(1)$ gauge group we get the best variational approximation in the transverse gauge $\partial_\mu A^\mu = 0$. That constraint is imposed by still taking the gauge-fixing term to be

$$\mathcal{L}_{fix} = \frac{1}{\epsilon}\left(\partial_\mu A^\mu\right)^2$$

(73)

where the limit $\epsilon \to 0$ is understood. This gauge choice is equivalent to a shift of the integration variables before the average, in order to cancel the longitudinal part of the gauge field A_μ. Then the average can be taken in Eq.(60) and, provided that $\mathcal{L}_{GEP}$ is even, the odd lagrangian terms give a vanishing contribution. Thus we can drop $\mathcal{L}_1$ and the odd terms of $\mathcal{L}_{YM}$ in the average, and the ghost term $\mathcal{L}_{gh}$ which does not contribute in the unitarity gauge. Insertion of Eq.(37) in the effective potential Eq.(60) yields

$$\mathcal{V}_{GEP}(\bar{\rho}) = -\log \int D_\rho e^{-\int d^4x \mathcal{L}_{GEP}} + \int d^4x \left\langle \mathcal{L}_{int} \right\rangle_\rho .$$

(74)

where the interaction Lagrangian now reads

$$\mathcal{L}_{int} = \mathcal{L}_\rho + \mathcal{L}_{fix} + \mathcal{L}_2 + \mathcal{L}_{YM}^{even} - \mathcal{L}_{GEP}$$

(75)

Next we take a shift of the scalar field ρ, and as usual[22] we define the scalar Higgs field h according to

$$h = \rho - \bar{\rho}.$$

(76)

A natural choice for the Gaussian trial Lagrangian is the sum of quadratic Gaussian Lagrangians for the gauge fields and the scalar Higgs field

$$\mathcal{L}_{GEP} = \mathcal{L}_{GEP}(h) + \mathcal{L}_{GEP}(W) + \mathcal{L}_{GEP}(Z) + \mathcal{L}_{GEP}(A)$$

(77)

with the Lagrangian terms defined according to

$$\mathcal{L}_{GEP}(h) = \frac{1}{2}\left(\partial_\mu h\right)^2 + \frac{1}{2}\Omega_h^2 h^2$$

(78)

$$\mathcal{L}_{GEP}(W) = \frac{1}{2}\left(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+\right)\left(\partial_\mu W_\nu^- - \partial_\nu W_\mu^-\right) + \Omega_W^2 W_\mu^+ W^{-\mu}$$

(79)

$$\mathcal{L}_{GEP}(Z) = \frac{1}{4}\left(\partial_\mu Z_\nu - \partial_\nu Z_\mu\right)^2 + \frac{1}{2}\Omega_Z^2 Z_\mu Z^\mu$$

(80)

$$\mathcal{L}_{GEP}(A) = \frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 + \frac{1}{\epsilon}\left(\partial_\mu A^\mu\right)^2 .$$

(81)

Here the masses Ω_h, Ω_W and Ω_Z must be regarded as variational parameters. With this choice we get $\langle h \rangle = 0$ and by the definition of h, Eq.(76), then $\langle \rho \rangle = \bar{\rho}$ as it was required in the derivation of the Gaussian effective potential Eq.(60). In order to evaluate $\mathcal{V}_{GEP}(\bar{\rho})$, according to Eq.(74) we also need $\mathcal{L}_{int}$ which now reads

$$
\mathcal{L}_{int} = V\left((\bar{\rho}+h)^2\right) - \frac{1}{2}\Omega_h^2 h^2 +
$$
$$
+ \left[\left(\frac{\bar{\rho}+h}{v}\right)^2 M_W^2 - \Omega_W^2 \right] W_\mu^+ W^{-\mu} + \frac{1}{2}\left[\left(\frac{\bar{\rho}+h}{v}\right)^2 M_Z^2 - \Omega_Z^2 \right] Z_\mu Z^\mu + \mathcal{L}_4 \tag{82}
$$

where $\mathcal{L}_4$ contains the quartic terms that come out from the product $(\vec{A}_\mu \times \vec{A}_\nu)^2$ in $\mathcal{L}_{YM}^{even}$

$$
\mathcal{L}_4 = e^2\left[(A_\mu A^\mu)(W_\nu^+ W^{-\nu}) - (W_\mu^+ A^\mu)(W_\nu^- A^\nu)\right] + e^2\frac{g^2}{g'^2}\left[(Z_\mu Z^\mu)(W_\nu^+ W^{-\nu}) - (W_\mu^+ Z^\mu)(W_\nu^- Z^\nu)\right] +
$$
$$
+ e^2\frac{g}{g'}\left[(W_\mu^+ A^\mu)(W_\nu^- Z^\nu) - 2(A_\mu Z^\mu)(W_\nu^+ W^{-\nu}) + (W_\mu^+ Z^\mu)(W_\nu^- A^\nu)\right] +
$$
$$
+ \frac{1}{2}g^2\left[(W_\mu^+ W^{-\mu})^2 - (W_\mu^+ W^{+\mu})(W_\nu^- W^{-\nu})\right]. \tag{83}
$$

The couplings can be written in terms of the mass parameters by the standard relations

$$
g^2 = \frac{4M_W^2}{v^2} \tag{84}
$$

$$
e^2 = \frac{4M_W^2}{v^2}\left(1 - \frac{M_W^2}{M_Z^2}\right) \tag{85}
$$

$$
\frac{g}{g'} = \frac{M_W}{\sqrt{M_Z^2 - M_W^2}}. \tag{86}
$$

However at this stage M_W and M_Z are just an alternative set of parameters and they are not physical masses.

The explicit evaluation of the Gaussian effective potential then follows by Wick's theorem through Eq.(74). As usual, the classical potential of the standard Higgs sector is written as

$$
V(\rho^2) = \frac{1}{2}m^2\rho^2 + \frac{1}{4!}\lambda\rho^4 \tag{87}
$$

and denoting by φ the average of the field ρ, $\varphi = \bar{\rho}$, a straightforward calculation yields the effective potential (GEP)

$$
\mathcal{V}_{GEP}(\varphi) = \frac{1}{2}m^2\varphi^2 + \frac{1}{2}m^2 I_0(\Omega_h) + \frac{\lambda}{4!}\varphi^4 + \frac{\lambda}{4}\varphi^2 I_0(\Omega_h) + \frac{\lambda}{8}\left[I_0(\Omega_h)\right]^2 - \frac{1}{2}\Omega_h^2 I_0(\Omega_h) +
$$
$$
+ I_1(\Omega_h) + 3I_1(\Omega_z) + 6I_1(\Omega_W) + I(\log\Omega_z + 2\log\Omega_W) +
$$
$$
+ \left[\frac{\varphi^2 + I_0(\Omega_h)}{4}g^2 - \Omega_W^2\right]J(\Omega_W) + \frac{1}{2}\left[\frac{\varphi^2 + I_0(\Omega_h)}{4}(g^2 + g'^2) - \Omega_Z^2\right]J(\Omega_Z) +
$$
$$
+ \left[\frac{9}{4}e^2 I_0(0) + \frac{3}{8}g^2 J(\Omega_W) + \frac{3}{4}\frac{e^2 g^2}{g'^2}J(\Omega_z)\right]J(\Omega_W) \tag{88}
$$

where the function $J(X)$ is

$$
J(X) = 3I_0(X) + \frac{I}{X^2} \tag{89}
$$

and the Euclidean integrals I, I_0, I_1 are defined according to

$$
I = \int_\Lambda \frac{d_E^4 k}{(2\pi)^4} \tag{90}
$$

$$
I_0(X) = \int_\Lambda \frac{d_E^4 k}{(2\pi)^4}\frac{1}{k^2 + X^2} \tag{91}
$$

$$
I_1(X) = \frac{1}{2}\int_\Lambda \frac{d_E^4 k}{(2\pi)^4}\log(k^2 + X^2). \tag{92}
$$

Here the symbol $\int_\Lambda$ means that the integrals are regularized by insertion of a cut-off Λ so that $k < \Lambda$: the Higgs sector is regarded as an effective model with a high energy scale Λ which plays the role of a further free parameter.

V. THE GAP EQUATIONS

The variational parameters Ω_h, Ω_W and Ω_Z must be determined by requiring that for any value of φ the GEP is at a minimum, thus the three parameters are implicit functions of the average of the field ρ. Once the parameters have been determined, the minimum point of $\mathcal{V}_{GEP}$ as a function of φ gives the vacuum expectation value of the field ρ. For any φ, the minimum of $\mathcal{V}_{GEP}$ is obtained by the constraints

$$\frac{\partial \mathcal{V}_{GEP}}{\partial \Omega_h^2} = \frac{\partial \mathcal{V}_{GEP}}{\partial \Omega_W^2} = \frac{\partial \mathcal{V}_{GEP}}{\partial \Omega_Z^2} = 0. \tag{93}$$

We find three coupled equations (gap equations) which define the implicit functions $\Omega_h(\varphi)$, $\Omega_W(\varphi)$ and $\Omega_Z(\varphi)$. Once the parameters have been set at their best value by solving the gap equations, the broken-symmetry vacuum expectation value of the field ρ takes the value φ_0 which is obtained by the vanishing of the total derivative

$$\frac{d\mathcal{V}_{GEP}}{d\varphi} = \frac{\partial \mathcal{V}_{GEP}}{\partial \varphi} + \sum_b \left(\frac{\partial \mathcal{V}_{GEP}}{\partial \Omega_b^2} \right) \left(\frac{d\Omega_b^2}{d\varphi} \right) \tag{94}$$

where the label b runs over the bosons W, Z and h. If the gap equations are satisfied then

$$\frac{\partial \mathcal{V}_{GEP}}{\partial \Omega_b^2} = 0 \tag{95}$$

and the total derivative is equal to the partial derivative. Then φ_0 follows from the vanishing of the simple partial derivative

$$\left(\frac{\partial \mathcal{V}_{GEP}}{\partial \varphi} \right)_{\varphi=\varphi_0} = 0. \tag{96}$$

Eq.(93) and Eq.(96) are a set of four coupled equations that give the phenomenological predictions of the model. Differentiating Eq.(88) the gap equations Eq.(93) can be written as

$$\Omega_h^2 = m^2 + \frac{\lambda}{2}\varphi^2 + \frac{\lambda}{2}I_0(\Omega_h) + \frac{g^2}{2}J(\Omega_W) + \frac{g^2 + g'^2}{4}J(\Omega_Z) \tag{97}$$

$$\Omega_Z^2 = (g^2 + g'^2)\frac{\varphi^2 + I_0(\Omega_h)}{4} + \frac{3e^2 g^2}{2g'^2}J(\Omega_W) \tag{98}$$

$$\Omega_W^2 = g^2\frac{\varphi^2 + I_0(\Omega_h)}{4} + \frac{3e^2 g^2}{4g'^2}J(\Omega_Z) + \frac{9}{4}e^2 I_0(0) + \frac{3}{4}g^2 J(\Omega_W). \tag{99}$$

The vacuum expectation value of the field ρ then follows from Eq.(96): the partial derivative reads

$$\frac{\partial \mathcal{V}_{GEP}}{\partial \varphi} = \varphi \left[m^2 + \frac{\lambda}{6}\varphi^2 + \frac{\lambda}{2}I_0(\Omega_h) + \frac{g^2}{2}J(\Omega_W) + \frac{g^2 + g'^2}{4}J(\Omega_Z) \right] \tag{100}$$

and insertion of Eq.(97) yields

$$\frac{d\mathcal{V}_{GEP}}{d\varphi} = \frac{\partial \mathcal{V}_{GEP}}{\partial \varphi} = \varphi \left[\Omega_h^2 - \frac{\lambda\varphi^2}{3} \right]. \tag{101}$$

Then Eq.(96) has two solutions: the unbroken symmetry stationary point $\varphi_0 = 0$ and the physical broken-symmetry vacuum expectation value

$$\varphi_0^2 = \frac{3}{\lambda}\Omega_h^2. \tag{102}$$

According, when φ_0 is set at its phenomenological value v, the self-coupling constant λ turns out to be proportional to the square of the mass parameter Ω_h, and a large Ω_h would not be compatible with perturbation theory. Conversely

the present variational calculation still holds for any large coupling, allowing for a full discussion of the Higgs sector. We notice that Ω_h is not the phenomenological mass M_h of the Higgs Boson which can be smaller than the variational parameter Ω_h. Here Ω_h may be regarded as the bare mass which appears in the zero-order Lagrangian $\mathcal{L}_{GEP}(h)$ in Eq.(78), and in principle it can be very large. The phenomenological mass of the Higgs boson arises from the curvature of the GEP at the broken-symmetry minimum. Streactly speaking we should also check that the curvature is positive, otherwise the solution of the coupled gap equations would not refer to a minimum of the GEP. At tree level, the perturbative result $M_h^2 = \lambda \varphi_0^2/3$ would be equivalent to Eq.(102) only if $\Omega_h = M_h$. In fact we will see that the bare mass Ω_h can be very large compared to the Higgs boson mass M_h, and even a light Higgs boson could be described by a strongly interacting Higgs sector with a very large self-coupling λ[25, 26].

The curvature of the GEP follows from the second derivative: from Eq.(101) we see that

$$\frac{d^2\mathcal{V}_{GEP}}{d\varphi^2} = \left[\Omega_h^2 - \frac{\lambda\varphi^2}{3}\right] + 2\varphi^2\left[\frac{d\Omega_h^2}{d\varphi^2} - \frac{\lambda}{3}\right] \tag{103}$$

At the unbroken symmetry stationary point $\varphi_0 = 0$ the second term vanishes

$$M_0^2 = \left(\frac{d^2\mathcal{V}_{GEP}}{d\varphi^2}\right)_{\varphi=0} = \Omega_h^2 \tag{104}$$

and the physical mass is $M_0 = \Omega_h$. Conversely in the phenomenological broken-symmetry vacuum the first term vanishes and the physical mass M_h is given by

$$M_h^2 = \left(\frac{d^2\mathcal{V}_{GEP}}{d\varphi^2}\right)_{\varphi=\varphi_0} = \frac{6\Omega_h^2}{\lambda}\left[\left(\frac{d\Omega_h^2}{d\varphi^2}\right)_{\varphi=\varphi_0} - \frac{\lambda}{3}\right]. \tag{105}$$

The derivatives of the variational parameters Ω_b can be obtained by differentiating the coupled gap equations Eq.(97),(98) and (99): we get the following set of coupled linear equations

$$\left[1 - \frac{\lambda}{2}\frac{\partial I_0(\Omega_h)}{\partial \Omega_h^2}\right]\left(\frac{d\Omega_h^2}{d\varphi^2}\right) - \frac{1}{4}(g^2 + g'^2)\frac{\partial J(\Omega_Z)}{\partial \Omega_Z^2}\left(\frac{d\Omega_Z^2}{d\varphi^2}\right) - \frac{g^2}{2}\frac{\partial J(\Omega_W)}{\partial \Omega_W^2}\left(\frac{d\Omega_W^2}{d\varphi^2}\right) = \frac{\lambda}{2} \tag{106}$$

$$\frac{g^2}{4}\frac{\partial I_0(\Omega_h)}{\partial \Omega_h^2}\left(\frac{d\Omega_h^2}{d\varphi^2}\right) + \left(\frac{3e^2g^2}{4g'^2}\right)\frac{\partial J(\Omega_Z)}{\partial \Omega_Z^2}\left(\frac{d\Omega_Z^2}{d\varphi^2}\right) - \left[1 + \frac{7g^2}{4}\frac{\partial J(\Omega_W)}{\partial \Omega_W^2}\right]\left(\frac{d\Omega_W^2}{d\varphi^2}\right) = -\frac{g^2}{4} \tag{107}$$

$$\frac{1}{4}(g^2 + g'^2)\frac{\partial I_0(\Omega_h)}{\partial \Omega_h^2}\left(\frac{d\Omega_h^2}{d\varphi^2}\right) - \left(\frac{d\Omega_Z^2}{d\varphi^2}\right) + \left(\frac{3e^2g^2}{2g'^2}\right)\frac{\partial J(\Omega_W)}{\partial \Omega_W^2}\left(\frac{d\Omega_W^2}{d\varphi^2}\right) = -\frac{1}{4}(g^2 + g'^2). \tag{108}$$

The solution is trivial, and insertion of $d\Omega_h^2/d\varphi^2$ in Eq.(105) yields the physical mass of the Higgs boson.

VI.   MODIFIED VARIATIONAL METHOD AND PHENOMENOLOGY

In order to explore the predictions of the model we have to face two major problems: we must find a consistent way to deal with the diverging integrals, and we must renormalize the bare parameters before any comparison with the phenomelogical observables can be made.

There are two very different approaches to the above problems. While a consistent and satisfactory renormalization scheme will be described in the next section, here we discuss a simpler path which can be seen as a modified low energy variational method. This approach relies on the opinion that the Higgs sector of the standard model is an effective model valid up to a physical energy cut-off Λ. Thus the parameters in the Lagrangian should be regarded as the physical effective values at that energy scale. Besides, the energy scale Λ could even be not too large compared to the physical masses. According, the integrals can be regularized by insertion of the cut-off, and the variational parameters Ω_b can be regarded as phenomenological masses (with the eventual renormalization of Ω_h arising from Eq.(105) above). Actually the existence of strongly diverging terms in the gap equations (mainly the J and I integrals) makes it obvious that Λ would give the scale of all the masses: in other words Λ could not be higher then ≈ 100 GeV. We could hardly find a physical meaning for such a small scale, but rather Λ should be regarded as a parameter that cuts the high energy effects in the loop integrals: that allows the variational method to describe the low energy physics

without having to depend too much on the high energy modes that usually spoil the predictive power of variational calculations in field theory[1]. In that sense Λ should not be regarded as a truly physical parameter, but rather as an internal parameter of the variational method. From a more formal point of view, that is equivalent to split the GEP as

$$\mathcal{V}_{GEP} = \mathcal{V}_{low} + \mathcal{V}_{high} \tag{109}$$

where $\mathcal{V}_{high}$ contains all the high energy contributions to $\mathcal{V}_{GEP}$, i.e. all the contributions that arise from integrations over $k > \Lambda$ in Eqs.(89),(90),(91),(92). We could define a modified variational method by taking the minimum of the low energy part $\mathcal{V}_{low}$ only. The resulting gap equations would be exactly the same as Eqs.(97),(98) and (99) but with a cut-off Λ in the diverging integrals. Thus a small cut-off can be regarded as a special choice of an optimized variational method which enhances the effects of low energy modes.

In this framework we may fix the masses at their known phenomenological value, and look for a set of values for the free parameters that satisfy the coupled gap equations. We can regard the masses $\Omega_W = 80.403 \pm 0.029$ GeV and $\Omega_Z = 91.1876 \pm 0.0021$ GeV as experimentally known physical masses[27]. From the Fermi constant $G_F = 1.16637 \cdot 10^{-5}$ GeV^{-2} the phenomenological weak coupling g of the effective lagrangian follows according to

$$\frac{g^2}{\Omega_W^2} = 4\sqrt{2}G_F \tag{110}$$

which yields $g = 0.6531$. The fine-structure constant at the weak scale[23] reads $e^2/4\pi = 1/128.87$ and determines the weak coupling $g' = 0.3555$ by Eq.(69). With these phenomenological inputs, we are left with an unknown mass, namely the Higgs boson mass parameter Ω_h, and the three free parameters λ, m^2, Λ that characterize the Higgs sector.

A first test of the model arises from a comparison of the known phenomenological energies v_W and v_Z, that we define as

$$\Omega_Z = \frac{1}{2} v_Z \sqrt{g^2 + g'^2} \tag{111}$$

$$\Omega_W = \frac{1}{2} g v_W. \tag{112}$$

The phenomenological data yield $v_W = 1/\sqrt{(\sqrt{2}G_F)} = 246.221$ GeV, $v_Z = 245.264$ GeV and $(v_W - v_Z)/v_W \approx (v_W^2 - v_Z^2)/(2v_W^2) = 3.88 \cdot 10^{-3} \pm 0.4 \cdot 10^{-3}$.

In the standard model, at tree-level both those energies are equal to the vacuum expectation value of the scalar field ρ. The small phenomenological difference arises from higher order corrections. In the present variational approximation the energies would differ according to the gap equations Eq.(98) and Eq.(99), and nor of them would be exactly equal to φ_0:

$$v_Z^2 = \varphi_0^2 + I_0(\Omega_h) + 6\cos^4\theta_W J(\Omega_W) \tag{113}$$

$$v_W^2 = \varphi_0^2 + I_0(\Omega_h) + 3\cos^2\theta_W J(\Omega_Z) + 9\sin^2\theta_W I_0(0) + 3J(\Omega_W) \tag{114}$$

where as usual we take $\cos^2\theta_W = e^2/g^2$ and $\sin^2\theta_W = e^2/g'^2$. The difference arises from one-loop contributions, and does not depend on the mass of the Higgs boson or on other parameters of the Higgs sector:

$$v_W^2 - v_Z^2 = 9\sin^2\theta_W I_0(0) + 3\cos^2\theta_W J(\Omega_Z) + 3(1 - 2\cos^4\theta_W)J(\Omega_W) \tag{115}$$

Of course this difference is sensitive to the magnitude of the cut-off Λ, and can be regarded as a phenomenological constraint on the cut-off. In the limit $\Lambda \to 0$ all the integrals in Eq.(115) vanish and we are left with the tree-level approximation $v_Z = v_W = \varphi_0$. In the limit $\Lambda \to \infty$ all the integrals diverge and the calculation has no pratical meaning.

Thus the difference in Eq.(115) can be regarded as a measure of the contribution of high energy modes, and the phenomenological requirement of keeping this difference at a small reasonable value is a strong constraint. In other words we can regard Λ as a free parameter and use Eq.(115) in order to establish its value by comparison with the phenomenological value of the difference $v_Z^2 - v_W^2$. In condensed matter[12, 13] this freedom has allowed a good fit of the experimental data, since the cut-off acts as a regulator of quantum fluctuations that can be scaled at the correct
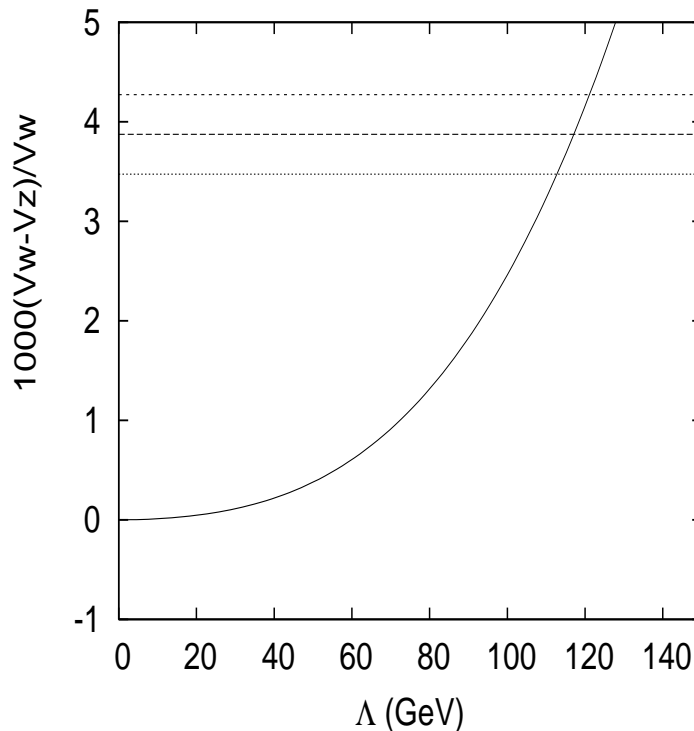
FIG. 1: The relative difference $(v_W - v_Z)/v_W \approx (v_W^2 - v_Z^2)/(2v_W^2)$ is reported as a function of the cut-off Λ according to Eq.(115). The average phenomenological value is reported as a dashed line, and it is expected to be confined between the dotted lines.

phenomenological value. However we do not expect that the parameter Λ might retain any physical role besides being a fit parameter, and its actual value cannot be taken too seriously as a physical energy scale. In fact, as expected, it turns out to be quite small, thus indicating that quantum fluctuations do not play a major role.

The relative difference is evaluated by Eq.(115) and reported in Fig.1 as a function of the cut-off Λ. It is an increasing function of Λ and it reaches the phenomenological value for $\Lambda \approx 115$ GeV. For larger values of Λ the quantum fluctuations add a large increasing contribution which eventually diverges. Then it is quite clear that, in this framework, the variational method does not introduce any dramatic change with respect to the tree-level description, since the role of quantum fluctuations is strongly suppressed by the small value of Λ. This result does not seem to be very satisfactory as according to Eq.(105) any renormalization of the Higgs Boson mass would be negligible and strongly suppressed by the small cut-off: in order to keep the difference between v_W and v_Z small, any interesting effect would be suppressed and the method would be equivalent to tree-level perturbation theory.

As the problem arises from the existence of strongly diverging integrals in finite phenomenological observables, and since those dievergences are known to be spurious (i.e. they cancel exactly at one-loop[28, 29]), we conclude that a satisfactory comparison with the phenomenological data requires a deeper and consistent renormalization of the physical observables.

VII. RENORMALIZATION AND PHENOMENOLOGICAL PREDICTIONS

In order to overcome the shortcomings of the previous section, we develop a consistent renormalization scheme which does not spoil the variational nature of the method in the Higgs sector, while retaining the structure of a standard perturbative calculation for the weak interactions. The variational results are regarded as the starting point of a standard perturbative renormalization process. This view had been suggested by Cea and Tedesco[30] who had also shown that the gaussian variational trial functionals can be regarded as an optimized variational basis. Higher order corrections can be derived by standard perturbative techniques in the variational basis, still retaining the variational nature of the calculation. In the zero-order Lagrangian $\mathcal{L}_{GEP}$, Eq.(77), the variational evaluation of the mass parameters Ω_b ensures that the residual interaction is minimal, and the so optimized perturbative theory should work better. We do not expect any important improvement in the weak sector where the couplings are very

small and the standard perturbative theory already works very well. However the method gives new insights on the Higgs sector, and on its non-perturbative range of strong coupling.

We take the view that all the variational parameters Ω_b must be regarded as bare parameters which define the bare optimized zero-order Lagrangian $\mathcal{L}_{GEP}$. Thus no comparison can be done with experimental masses before renormalization. The cut-off Λ is regarded as a large energy scale of the effective model, and the bare masses Ω_b are derived by solution of the gap equations Eq.(97),(98),(99). These bare masses would be very large, typically of order Λ, and must be renormalized. For the weak sector we can use the standard one-loop renormalization of masses, while for the Higgs sector a consistent non-perturbative renormalization is required, and it is provided by the curvature of the GEP Eq.(105) which contains non-perturbative contributions to the mass shift. Actually in the simple scalar theory Eq.(105) can be shown to be equivalent to the sum of bubble diagrams to all orders[26, 31]. Thus the shape of the GEP contains non-perturbative information on the renormalization of the bare parameters. In the Higgs sector it is quite important that we rely on a non-perturbative renormalization method as we would like to discuss some untrivial features of the strong coupling regime.

In the weak sector all divergences are known to cancel at one-loop even in the unitarity gauge[28, 29] and the resulting perturbative corrections have been reported to be very small, as it should be for any perturbative correction arising from weak couplings. We can recover the same results from the optimized zero-order Lagrangian $\mathcal{L}_{GEP}$ with bare masses Ω_b that are solution of the gap equations and with φ set at the minimum of the GEP $\varphi = \varphi_0 = v$ which we expect to be the phenomenological value. The one-particle irreducible one-loop self-energy reads

$$\Sigma_b^{(1L)} = \Omega_b^2 - M_b^2 + \Sigma_b^{(1)} + \Sigma_b^{(2)} \tag{116}$$

where the index b runs over the two bosons W and Z, while $\Sigma^{(1)}$ and $\Sigma^{(2)}$ are the first and second order contributions to the one-loop self-energy, and the masses M_b are the standard tree level masses as reported in Eqs.(71),(72). It can be easily shown that the gap equations Eqs.(98),(99) can be written in terms of the first order self energy as

$$\Omega_b^2 = M_b^2 - \Sigma_b^{(1)} \tag{117}$$

and insertion in Eq.(116) shows that $\Sigma_b^{(1L)} = \Sigma_b^{(2)}$ which is a general property of the GEP. At one-loop the renormalized mass is

$$(\Omega_b^2)_R = \Omega_b^2 - \Sigma_b^{(1L)} \tag{118}$$

and again, insertion of the gap equation Eq.(117) yields

$$(\Omega_b^2)_R = M_b^2 - \Sigma_b^{(1)} - \Sigma_b^{(2)}. \tag{119}$$

Formally this is exactly the same result which we would obtain by one-loop renormalization of the standard model lagrangian with bare masses M_b^2 (apart from the choice of the free propagator in the self energy, which in the present calculation contains the bare masses Ω_b). The sum of all one-loop terms contributing to $\Sigma_b^{(1)} + \Sigma_b^{(2)}$ is known to be finite and very small[28, 29] compared to M_b^2. Thus we get the standard one-loop result $(\Omega_b^2)_R \approx M_b^2$ up to small perturbative corrections. We can say that the one-loop renormalization of the bare variational masses allows us to recover the experimental phenomenology for the gauge bosons. The result would be trivial were it not for the Higgs sector where the coupling cannot be assumed to be small and where the above perturbative renormalization would not be reliable. In the Higgs sector the variational mass parameter Ω_h depends on the self-coupling λ and on the vacuum expectation value $\varphi_0 = v$ through the minimum condition Eq.(102) which gives to Ω_h a clear physical phenomenological meaning: Ω_h sets the scale of the self-coupling λ which reads

$$\lambda = \frac{3\Omega_h^2}{v^2} \tag{120}$$

Here we do not have any problem at insuring that Ω_h, the solution of the gap equation Eq.(97), takes a finite phenomenological value: in fact the existence of the free mass parameter m^2 makes sure that the solution of the gap equation Eq.(97) can be any number we like. Thus we fix m^2 in order to satisfy the minimum condition Eq.(120) and take Ω_h as a free parameter which gives the strength of the self-coupling λ. We do not need to deal with infinities, but we must address the problem of mass renormalization anyway, as the residual interaction shifts the physical Higgs mass that cannot be taken to be equal to the variational mass parameter Ω_h. In fact we have seen that the shape of the GEP contains non-perturbative effects which can be shown to be the sum of bubble diagrams to all orders[26, 31]. A non-trivial mass renormalization comes from the curvature of the GEP which allows us to take the physical mass of the Higgs boson M_h according to Eq.(105). At this stage there is no reason why the cut-off Λ should be small,
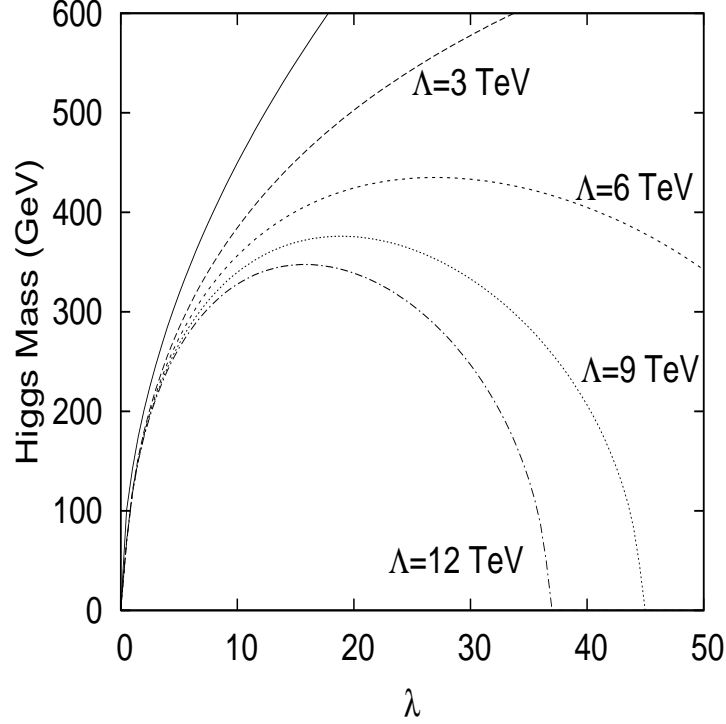
FIG. 2: The Higgs mass M_h according to Eq.(105) as a function of the self-coupling parameter λ and for several choices of the cut-off Λ ranging from 3 TeV to 12 TeV (broken lines). The solid line is the tree-level result $M_h = \Omega_h$ ($\Lambda = 0$).

since all the phenomenological observables are finite any way. We assume that Λ is some very large energy scale and examine the behaviour of the physical mass M_h as a function of the self-coupling λ.

As shown in Fig.2 the physical mass M_h is not a monotonous increasing function of the coupling, but it reaches a maximum and then decreases. M_h^2 eventually becomes negative at some large coupling, indicating that the broken-symmetry solution becomes unstable. We get an upper bound for the coupling and, before reaching it, a low mass non-perturbative strong coupling range. In this scenario a light Higgs can be found for a small coupling (perturbative light Higgs) but also for a large coupling (non-perturbative light Higgs). A very strong self-coupling reduces the mass: this effect cannot be predicted by any perturbative calculation. Moreover the reduction of mass increseas with the increasing of the cut-off Λ and eventually an infinite cut-off would make the Higgs boson mass vanishing: the broken symmetry vacuum would become unstable for any coupling as the upper bound of λ would go to zero. That is in agreement with the well known triviality of the scalar theory which requires the existence of a large but finite cut-off. In Fig.2 the tree-level approximation $M_h = \Omega_h$ is also reported for comparison: it is equivalent to the variational calculation for a very small cut-off Λ as discussed in the previous section. We can see that in the perturbative regime of small λ the Higgs mass is almost insensitive to the size of the cut-off, and the perturbative predictions agree with the variational result: the mass increases as the square root of the self-coupling λ. Conversely, in the strong coupling regime the mass of the Higgs boson depends on the size of the cut-off and becomes very small compared to the perturbative prediction which cannot be trusted any more. For instance at $\Lambda = 12$ TeV, a relatively light Higgs boson with $M_h \approx 200$ GeV is predicted for $\lambda \approx 2.5$ (perturbative weak-coupling range) but also for $\lambda \approx 40$ (non-perturbative strong-coupling range). The mass is the same in both cases but we expect a different behaviour for the scattering amplitudes in the strong-coupling range[26].

The prediction of a light Higgs boson in the strong coupling regime had been discussed in simplified models which neglected the gauge interactions[25, 26] and in the Abelian gauge interacting U(1) theory[19]. Here we confirm the same trend in the framework of the full SU(2)×U(1) gauge theory.

In Fig.3 the prediction of the GEP for a simple scalar theory[25, 31] is reported for comparison. As expected, the effect of gauge interactions is very small and can be neglected for a qualitative discussion of the Higgs sector.

Quite interesting the physics of the Higgs sector changes according to the choice of the cut-off Λ with a cross over point at $\Lambda \approx 3.7$ TeV separating the "small cut-off" scenario from the "large cut-off" one. As shown in Fig.4 the Higgs boson mass is an increasing function of the self-coupling λ for $\Lambda < 3.5$ TeV. At $\Lambda \approx 3.7$ TeV a minimum appears and deepens until $\Lambda \approx 3.9$ Tev where the minimum of M_h^2 becomes negative and a gap of prohibited couplings opens.
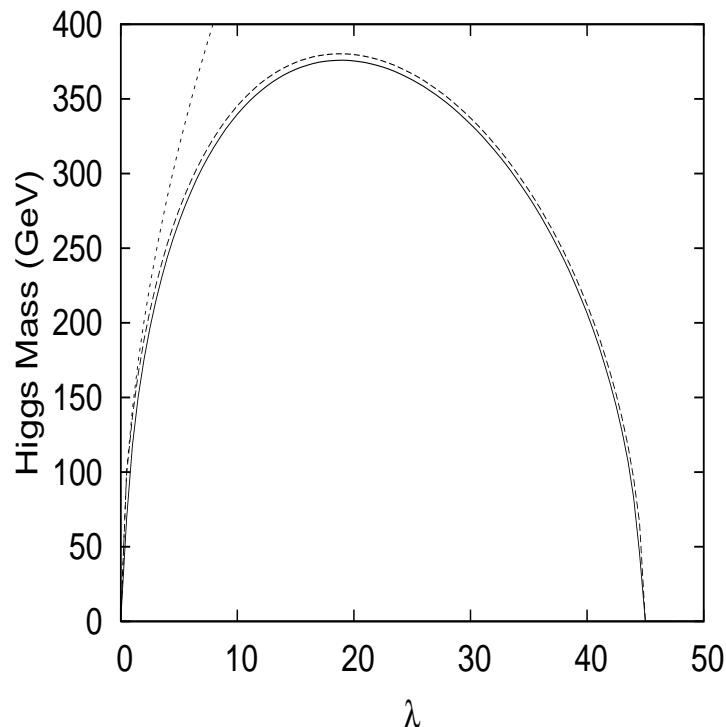
FIG. 3: The Higgs mass M_h according to Eq.(105) as a function of the self-coupling parameter λ for $\Lambda = 9$ TeV (solid line). The dashed line is the simple scalar theory result[25, 31]. For comparison the tree-level result $M_h = \Omega_h$ ($\Lambda = 0$) is reported as a dotted line.

We notice that the existence of a similar cross-over had been observed at $\Lambda \approx 3$ Tev in a quite different variational calculation for the scalar theory[26]. Thus it seems to be a genuine feature of the standard model which cannot be shown by any perturbative approximation.

In the "large cut-off" scenario, say for $\Lambda > 3.9$ TeV, there is a gap in the allowed range of λ, and this gap increases with the increasing of the cut-off. In this scenario a light Higgs boson, with a very small mass, can be compatible with three different couplings: for instance at $\Lambda \approx 3.9$ TeV (as shown in Fig.4) these are $\lambda_1 \rightarrow 0$ (perturbative solution), $\lambda_2 \approx 230$ and $\lambda_3 \approx 520$. Of course any other choice for Λ would provide a different set of couplings for the same Higgs mass. For a very large cut-off Λ the bigger coupling λ_3 becomes very large and probably has no physical relevance, while the intermediate coupling λ_2 becomes smaller and would describe a strongly coupled Higgs sector. The general behaviour is shown in Fig.2 where at $\Lambda = 12$ TeV the intermediate coupling reduces to $\lambda_2 \approx 37$. Thus for a large enough cut-off Λ we can predict the possible existence of a light Higgs boson with a large but reasonable self-coupling λ_2.

The possible existence of a light Higgs boson with a very strong self-interaction seems to be a non-perturbative feature of the standard model[19, 25, 26]. Thus the eventual experimental finding of a light Higgs mass $M_h \approx 200$ TeV would not rule out a strongly interacting Higgs sector. However we expect that a strongly interacting light Higgs boson should show a different behaviour when compared with the perturbative predictions: scattering amplitudes should be different and should tell us about the real strength of the self-coupling[32].

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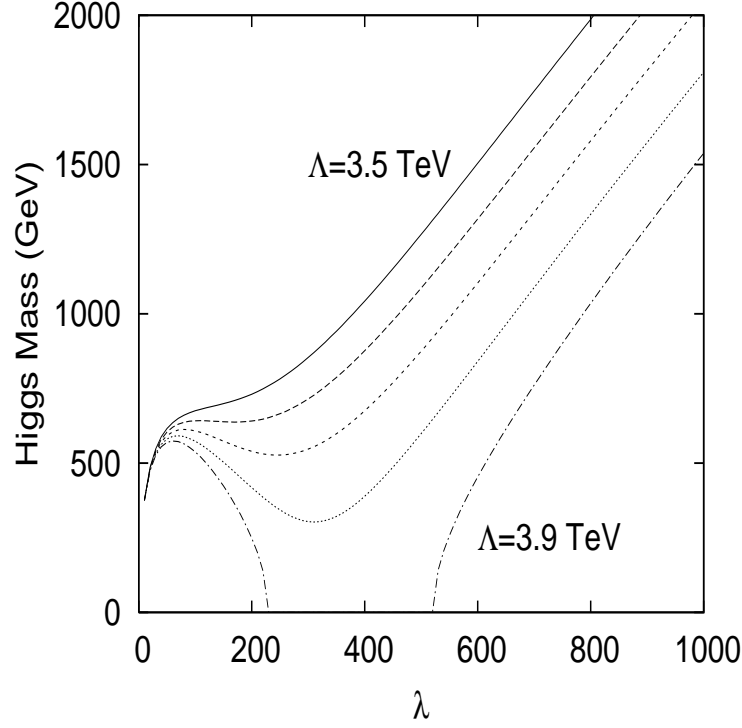[7] M. Weinstein, S. Drell, and S. Yankielowicz, Phys. Rev. D **14**, 487 (1976).

FIG. 4: The Higgs mass M_h according to Eq.(105) as a function of the self-coupling parameter λ for Λ=3.5, 3.6, 3.7, 3.8 and 3.9 TeV.

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